Sets

A well defined collection of objects. Number of objects can be finite or infinite, but it has to be well defined, i.e. if an element is given, the definition should be able to properly distinguish, whether it belongs to the set or not.

For example

- A set of natural numbers is well defined, and hence a set.
- A set of intelligent students of your class, is not well defined, hence not a set.

Representation of sets

There are two forms for representation of sets.

I. **Roaster form or tabular form**: When elements belonging to the set are listed, like A = {1, 2, 3, 4}.

II. **Set builder from**: When the differentiation of elements of the sets is given, like A = [{x : x belongs to N and x $\leq$ 4}.

Types of Sets

- **Empty or Null Set**: Represented as {} or ϕ, it is a set with no elements in it.
- **Singleton Set :** A set with just one element.
- **Finite and Infinite Sets :** A set with finite number of elements is a finite set, else its infinite set.
- **Cardinal number (or order) of a finite set**: The number of elements in a set is represented by cardinal number. So number of elements in set A is written as n (A) which is the cardinal number of set A.

- **Equal sets**: Sets with same elements in them are called equal sets. Note that, in a set, the number of times an element is repeated is irrelevant and the order in which the elements occur is also not important.
- **Subsets** : Set A is subset of set B, if every element of A belongs to B and is written as $A \subseteq B$ or $B \supseteq A$ and read as A is a subset of B or B is a superset of A, respectively.

  Note: If $A \subseteq B$ and $A \neq B$, we write it as $A \subset B$ and is called a proper subset. Every set is a subset and superset of itself. ϕ is a subset of every set. If $n(A) = n$, then number of subsets of A is 2^n and number of proper subsets is $2^n - 1$ (excluding set A itself).

  Power sets: Set of all possible subsets of set A is called power set of A and represented as P (A). Number of elements in P (A). as discussed above, is 2^n.

Operations on Sets

Union of two sets: Represented as $A \cup B$, and is set of all elements of A and B taken together.

So $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$x \in A \cup B \Rightarrow x \in A$ <u>or</u> $x \in B$

$x \notin A \cup B \Rightarrow x \notin A$ <u>and</u> $x \notin B$.

Intersection of two sets: Represented as $A \cap B$, it is the set of all elements which belong to both A and B

So, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$x \in A \cap B \Rightarrow x \in A$ <u>and</u> $x \in B$

$x \notin A \cap B \Rightarrow x \notin A$ <u>or</u> $x \notin B$.

If $A \cap B = \phi$, A and B are called disjoint sets.

Difference of two sets: A–B $\Rightarrow$ set of elements of A which are not present in B.

So $A - B = \{x: x \in A \text{ and } x \notin B\}$

$A - B \subseteq A$ and $B - A \subseteq B$, $A - B \neq B - A$

$A - B$, $B - A$ and $A \cap B$ are disjoint sets and together they form $A \cup B$.

Symmetric difference: Denoted as $A\triangle B$ and is defined as $(A - B) \cup (B - A)$.

$A \triangle B$ is also equal to $(A \cup B) - (A \cap B)$

Complement of a set: Complement of a set : if U is the universal set and A is subset of U, then A' read as A complement, is U–A. So $x \in A' \Rightarrow x \notin A$ and $x \in U$.

Some terminology

Idempotent law

for a set A, $A \cup A = A$ and $A \cap A = A$.

$A \cup \phi = A$, $A \cap \phi = \phi$ and $A \cup U = U$ and $A \cap U = A$

Commutative Law

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associative Law

$A \cup (B \cup C) = (A \cup B) \cup C$ and

$A \cap (B \cap C) = (A \cap B) \cap C$.

Distributive Law

For any three sets A, B, C,

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

And $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

$(A')' = A$

De Morgan's law:

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

$A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cap C) = (A - B) \cup (A - C)$

A and B are subsets of $A \cup B$ and $A \cap B$ is subset of both A and B.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Relations

Cartesian Product

Represented by

$A \times B$, $A \times B = \{x,y\} : x \in A$ and $y \in B$

Note

If A or B is ϕ, then $A \times B = \phi$

$A \times B \neq B \times A$

If $n(A) = n$, $n(B) = m$, then $n(A \times B) = mn$.

Some results related to Cartesian product of sets

i. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

ii. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

iii. $A \times (B - C) = (A \times B) - (A \times C)$

iv. $A \times B = B \times A$ iff $A = B$

Relation

- A relation R, from A to B is defined as a subset of $A \times B$. So $R = \{(a, b) : (a, b) \in A \times B\}$.
- Every subset of $A \times B$ is a relation from A to B. If the relation is from A to A, we say that R is a relation on set A and hence R is a subset of $A \times A$. If a is related to b, we write $(a, b) \in R$ or a R b.

  If $n(A) = n$, $n(B) = m$, then $n(A \times B) = mn$, so number of relation from A to B is equal to number of subsets of $A \times B$ which is 2^{mn}.

Inverse of Relation

- If R is a relation from A to B. Then R^{-1} is inverse relation and is defined as,

  $R^{-1} = \{(a, b) : (b, a) \in R\}$.

- If R is a relation from A to B i.e. $R : A \to B$, then domain of R =

  $\{a : a \in A \text{ and } (a, b) \in R \text{ for some } b \in B\}$.

- Range of R =

  $\{b : b \in B \text{ and } (a, b) \in R \text{ for some } a \in A\}$.

  Domain of R = Range of R^{-1} and Range of

  R = Domain of R^{-1}.

Types of Relation

Let A be a non-empty set and R be a relation on A, then

- **Reflexive Relation** : R is reflexive if $\forall$. $a \in A \Rightarrow (a, a) \in R$.
- **Symmetric Relation** : R is symmetric, if $\forall\ (a, b) \in R \Rightarrow (b, a) \in R$.
- **Transitive Relation** : R is transitive if $\forall\ (a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.
- **Equivalence Relation** : A relation which is reflexive, symmetric and transitive, is an equivalence relation.

Function

Introduction

If A and B are two non-empty sets, then a rule *f* which associated to each $x \in A$ a unique number

$y \in B$ is called a function from *A* to *B* and we write, $f: A \to B$.

Some Important Definitions

1. **Real numbers:**

   Real numbers are those which are either rational or irrational. The set of real numbers is denoted by *R*.

   i. **Rational numbers:**

      All numbers of the form p / q where *p* and *q* are integers and $q \neq 0$, are called rational numbers and their set is denoted by *Q.*

      e.g. 0.4, 5/3 are rational numbers.

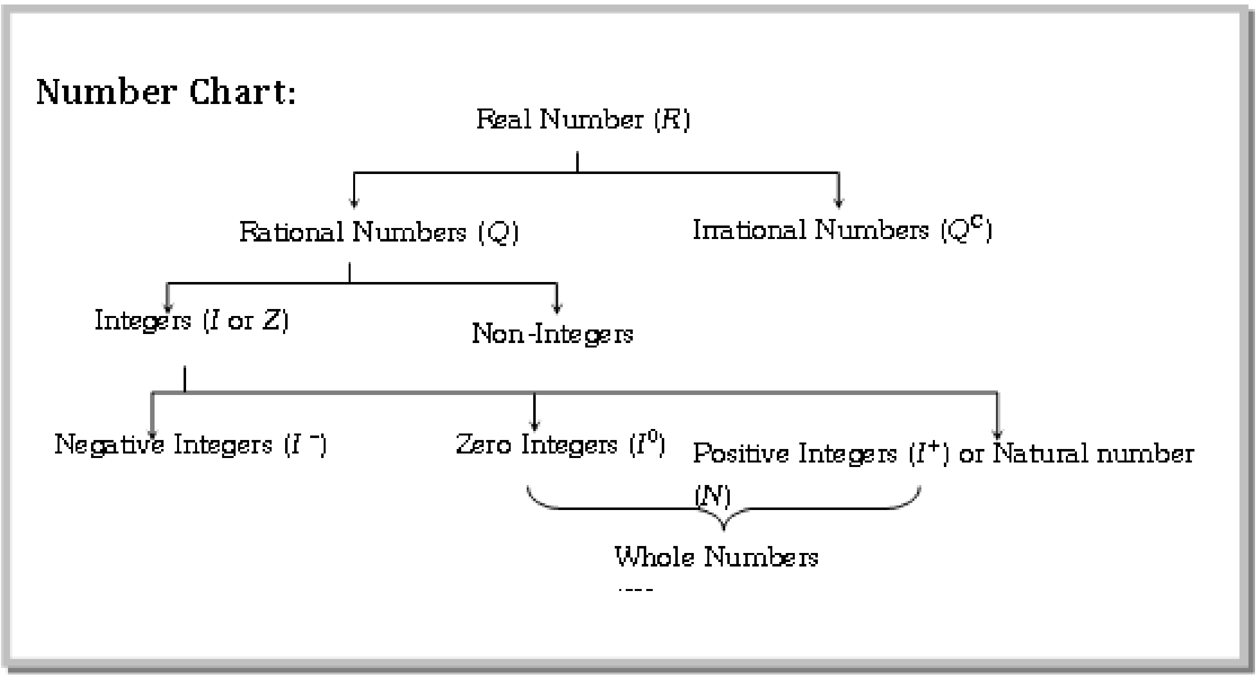
   ii. **Irrational numbers**

      Those are numbers which cannot be expressed in form of p / q are called irrational numbers and their set is denoted by Q^c (i.e., complementary set of Q) *e.g.* $\sqrt{2}$, $1-\sqrt{5}$ are irrational numbers.

   iii. **Integers**

      The numbers ....− 3, − 2, − 1, 0, 1, 2, 3, …….. are called integers. The set of integers is denoted by *I* or *Z*. Thus, *I* or

      $Z = \{\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$



- Set of positive integers = {1, 2, 3, ...}
- Set of negative integers = {- 1, – 2, – 3, ...}.
- Set of non negative integers ={0, 1, 2, 3, ..}
- Set of non positive integers =

  {0, – 1, – 2, – 3,.....}
- Positive real numbers: $R^+ = (0, \infty)$
- Negative real numbers: $R^- = (-\infty, 0)$
- R_0: all real numbers except 0 (Zero)
- Imaginary numbers: C= {I, ω, ...}
- Even numbers: E = { 0, 2, 4, 6, ...}
- Odd numbers: O = {1, 3, 5, 7, ...}
- Prime numbers : The natural number greater

  than 1 which is divisible by 1 and itself only is called prime number.
- In rational numbers, the digits after decimal

  either terminate or start recurring (repeating)
- 0 (zero) is a rational number

- In irrational numbers, digits after decimal neither terminate not recur
- π and e are called special irrational quantities
- ∞ is not a number, so it is neither a rational number nor an irrational number

2. **Related quantities**

When two quantities are such that the change in one is accompanied by the change in other, *i.e.,* if the value of one quantity depends upon the other, then they are called related quantities. *e.g.* the area of a circle ($A = \pi r^2$) depends upon its radius (r) as soon as the radius of the circle increases (or decreases), its area also increases (or decreases). In the given example, A and r are related quantities.

3. **Variable**

A variable is a symbol which can assume any value out of a given set of values. The quantities, like height, weight, time, temperature, profit, sales etc, are examples of variables. The variables are usually denoted by x, y, z, u, v, w, t etc. There are two types of variables mainly:

i. **Independent variable** : A variable which can take any arbitrary value, is called independent variable.

ii. **Dependent variable** : A variable whose value depends upon the independent variable is called dependent variable. *e.g.* $y = x^2$ if $x = 2$ then $y = 4 \Rightarrow$so value of y depends on x. y is dependent and x is independent variable here.

4. Constant

A constant is a symbol which does not change its value, *i.e.*, retains the same value throughout a set of mathematical operation. These are generally denoted by *a*, *b*, *c* etc. There are two types of constant.

i. **Absolute constant** : A constant which remains the same throughout a set of mathematical operation is known as absolute constant. All numerical numbers are absolute constants, *i.e.* 2, $\sqrt{3}$, e, π etc. are absolute constants.

ii. **Arbitrary constant** : A constant which remains same in a particular operation, but changes with the change of reference, is called arbitrary constant *e.g.* y = mx + c represents a line.

Here *m* and *c* are constants, but they are different for different lines. Therefore, *m* and *c* are arbitrary constants.

5. Absolute value

The absolute value of a number *x*, denoted by |*x*|, is a number that satisfies the conditions $|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0. \\ x & \text{if } x > 0 \end{cases}$

We also define |*x*| as follows, |*x*|= maximum {*x*, – *x*} or $|x| = \sqrt{x^2}$

The properties of absolute value are

i. The inequality $|x| \leq a$ means $-a \leq x \leq a$

ii. The inequality $|x| \geq a$ means $x \geq a$ or $x \leq -a$

iii. $|x \pm y| \leq |x| + |y|$ and $|x \pm y| \geq |x| - |y|$

iv. $|xy| = |x|\,|y|$

v. $|x/y| = |x|/|y|$ where $|y| \neq 0$

6. Greatest integer

Let $x \in R$ Then $[x]$ denotes the greatest integer less than or equal to x; *e.g.*

$[1.34] = 1$, $[-4.57] = -5$, $[0.69] = 0$ etc.

7. Fractional part

We know that $x \geq [x]$. The difference between the number 'x' and its integral value '$[x]$' is called the fractional part of x and is symbolically denoted as $\{x\}$. Thus, $\{x\} = x - [x]$

e.g., if $x = 4.92$ then $[x] = 4$ and $\{x\} = 0.92$.

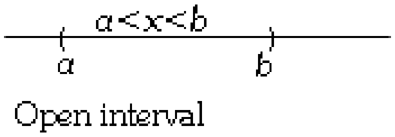
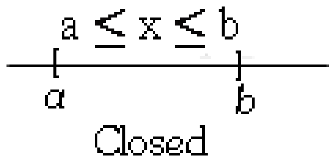
- Fractional part of any number is always

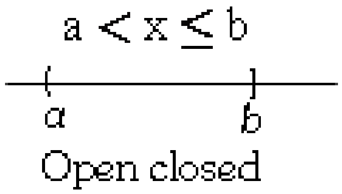
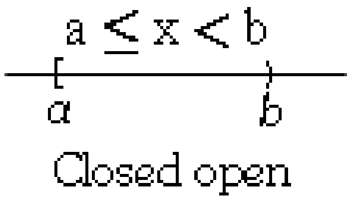
  non – negative and less than one.

Intervals

If a variable x assumes any real value between two given numbers, say a and b ($a < b$) as its value, then x is called a continuous variable. The set of real numbers which lie between two specific numbers is called the interval.

There are four types of interval:

1. **Open interval:**	2. **Closed interval:**
Let *a* and *b* be two real numbers such that a <b, then the set of all real numbers lying strictly between *a* and *b* is called an open interval and is denoted by]*a*, *b*[or (*a*, *b*). Thus,]*a*, *b*[or (*a*, *b*) = {x ∈R : a < x < b}  	Let *a* and *b* be two real numbers such that *a* < *b*, then the set of all real numbers lying between *a* and *b* including *a* and *b* is called a closed interval and is denoted by [*a*, *b*]. Thus, [*a*, *b*] = {x ∈ R : a ≤ x ≤ b} 

2. **Open-Closed interval :**	3. **Closed-Open interval:**
It is denoted by]*a*, *b*] or (*a*, *b*] and]*a*, *b*] or (*a*, *b*] = {x ∈ R : a < x ≤ b}  	It is denoted by [*a*, *b*[or [*a*, *b*) and [*a*, *b*[or [*a*, *b*) = {x ∈ R : a ≤ x < b} 

Definition of Function

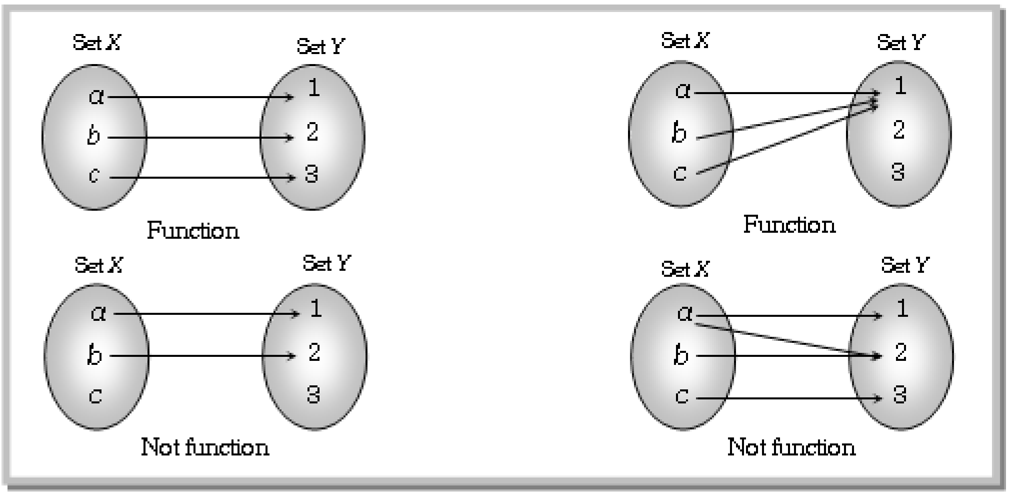
1. Function can be easily defined with the help of the concept of mapping. Let *X* and *Y* be any two non-empty sets. "A function from *X* to *Y* is a rule or correspondence that assigns to each element of set *X*, one and only one element of set *Y*".

Let the correspondence be '*f*' then mathematically we write f : X → Y where y = f(x), x ∈ X and y ∈ Y We say that '*y*' is the image of '*x*' under *f* (or *x* is the pre image of *y*).

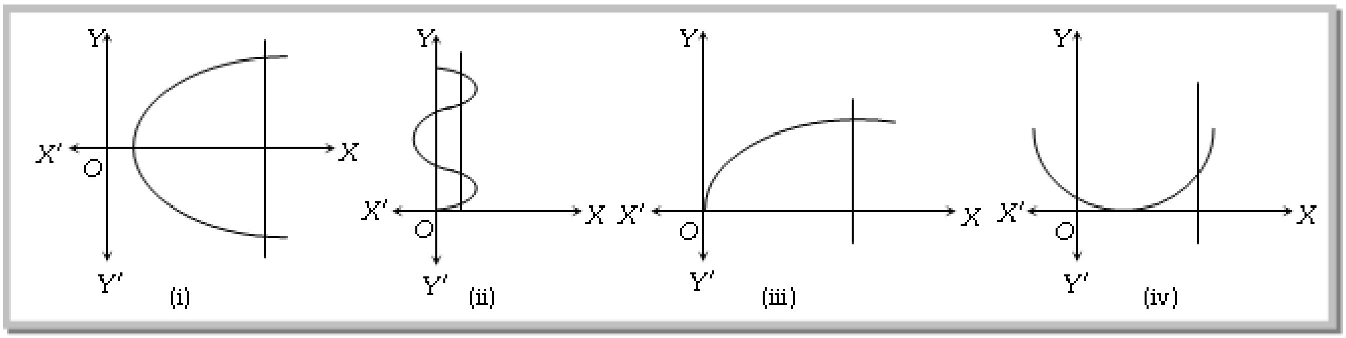
Two things should always be kept in mind:

i. A mapping f: X → Y is said to be a function if each element in the set *X* has its image in set *Y*. It is also possible that there are few elements in set *Y* which are not the images of any element in set *X*.

ii. Every element in set *X* should have one and only one image. That means it is impossible to have more than one image for a specific element in set *X*. Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from *X* and *Y*) *e.g.*



2. Testing for a function by vertical line test : A relation f: A → B is a function or not it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to *Y*-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.



3. **Number of functions** : Let *X* and *Y* be two finite sets having *m* and *n* elements respectively. Then each element of set *X* can be associated to any one of *n* elements of set *Y*. So, total number of functions from set *X* to set *Y* is n^m.

4. **Value of the function** : If y = f(x) is a function then to find its values at some value of *x*, say x = a we directly substitute $x = a$ in its given rule f (x) and it is denoted by f (a).

   E.g. If $f(x) = x^2 + 1$, then $f(1) = 1^2 + 1 = 2$,

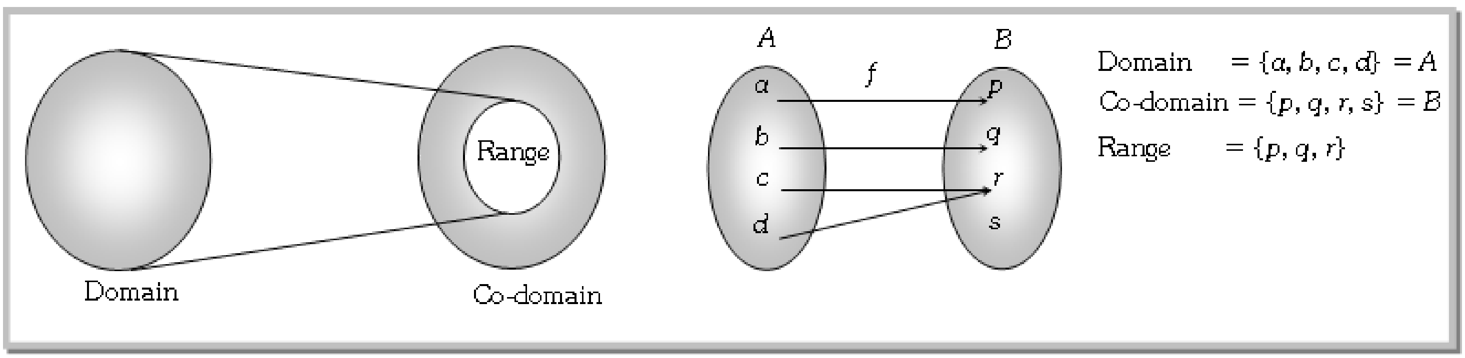
   $f(2) = 2^2 + 1 = 5$, $f(0) = 0^2 + 1 = 1$ etc.

Domain, Co-domain and Range of Function

If a function *f* is defined from a set of *A* to set *B* then for $f: A \to B$ set *A* is called the domain of function *f* and set *B* is called the co-domain of function *f*. The set of all *f*-images of the elements of *A* is called the range of function *f*.

In other words, we can say Domain = All possible values of *x* for which *f*(*x*) exists.

Range = For all values of *x*, all possible values of *f*(*x*).



1. Methods for finding domain and range of function

i. Domain

Expression under even root

(*i.e.*, square root, fourth root etc.) ≥ 0

Denominator $\neq 0$

If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x) \,.\, g(x)$ is $D_1 \cap D_2$.

While domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

Domain of $\sqrt{f(x)}$ is D_1 such that $f(x) \geq 0$.

Algebra of Functions

Let f(x) and g(x) be two real and single-valued functions, with domains X_f , X_g and ranges Y_f and Y_g respectively. Let $X = X_f \cap X_g \neq \phi$. Then, the following operations are defined.

1. Scalar multiplication of a function

$(c, f)(x) = c\, f(x)$ where c is a scalar. The new function $c\, f(x)$ has the domain X_f.

2. Addition/subtraction of functions

$(f \pm g)(x) = f(x) \pm g(x)$. The new function has the domain X.

3. Multiplication of functions

$(fg)(x) = (gf)(x) = f(x)\, g(x)$. The product function has the domain X.

4. Division of functions

i. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The new function has the domain X, except for the values of x for which $g(x) = 0$.

ii. $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$. The new function has the domain *X*, except for the values of *x* for which f(x) = 0.

5. Equal functions

Two function *f* and *g* are said to be equal functions, if and only if

i. Domain of *f* = domain of *g*

ii. Co-domain of *f* = co-domain of *g*

iii. f(x) = g(x) ∀ x ∈ their common domain

6. Real valued function

If *R*, be the set of real numbers and *A*, *B* are subsets of *R*, then the function f: A → B is called a real function or real -valued function.

Kinds of Function

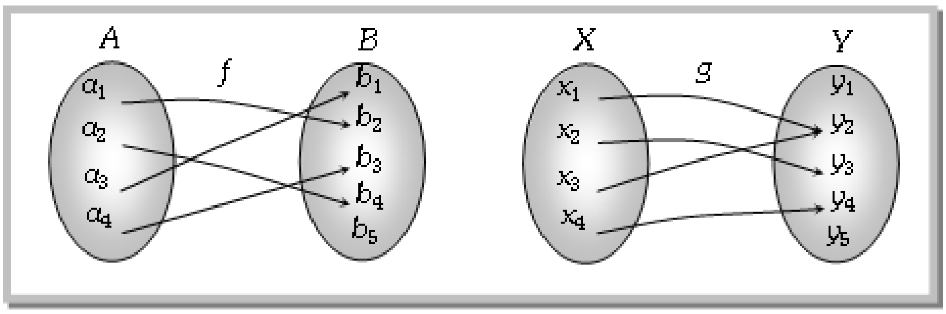
1. One-one function (injection)

A function f: A → B is said to be a one-one function or an injection, if different elements of *A* have different images in *B*. Thus, f: A → B is one-one.

⇔ a ≠ b ⇒ f(a) ≠ f(b) for all a, b ∈ A ⇔ f(a)

= f(b) ⇒ a = b for all a, b ∈ A.

E.g. Let f: A → B and g: X → Y be two functions represented by the following diagrams.



Clearly, f: A → B is a one-one function.

But g: X → Y is not one-one function because two distinct elements x_1 and x_3 have the same image under function *g*.

i. Method to check the injectivity of a function

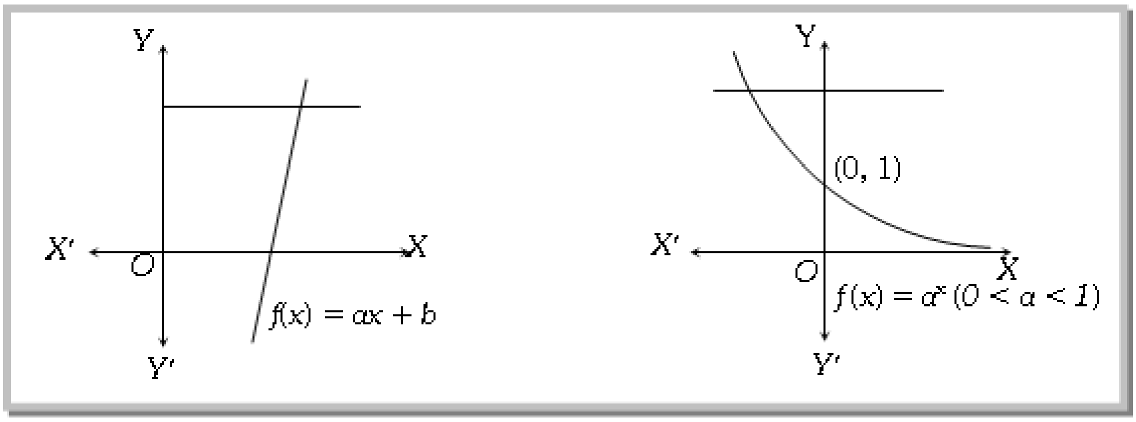
Step I: Take two arbitrary elements *x*, *y* (say) in the domain of *f.*

Step II: Put f(x) = f(y).

Step III : Solve f(x) = f(y). If f(x) = f(y) gives *x* = *y* only, then f: A → B is a one-one function (or an injection). Otherwise not.

If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

If the graph of the function y = f(x) is given and each line parallel to *x*-axis cuts the given curve at maximum one point then function is one-one. *e.g.*



ii. Number of one-one functions (injections) :

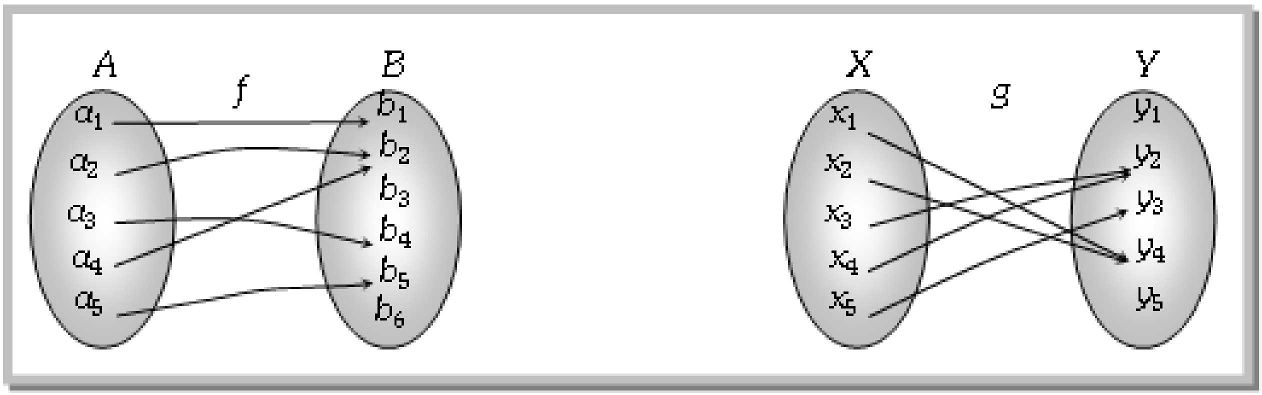
If *A* and *B* are finite sets having *m* and *n* elements respectively, then number of one-one functions from *A* to *B* $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

2. Many-one function

A function f: A → B is said to be a many-one function if two or more elements of set *A* have the same image in *B*.

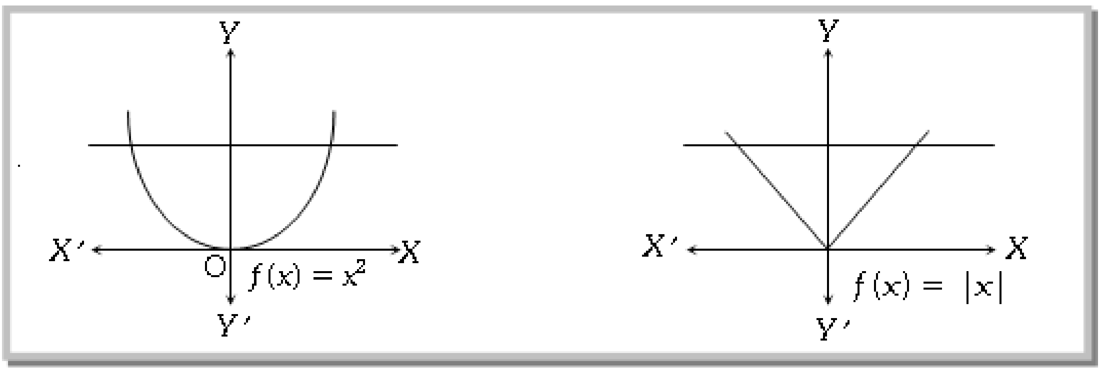
Thus, f: A → B is a many-one function if there exist x, y ∈ A such that x ≠ y but f(x) = f(y).

In other words, f: A → B is a many-one function if it is not a one-one function.



If function is given in the form of set of ordered pairs and the second element of at least two ordered pairs are same then function is many-one.

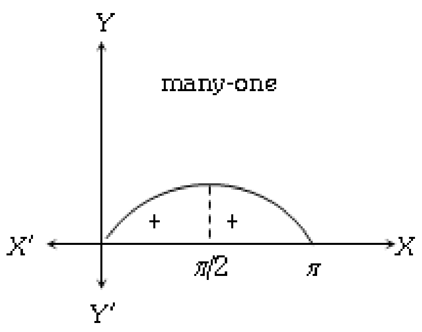
If the graph of y = f(x) is given and the line parallel to *x*-axis cuts the curve at more than one point then function is many-one.



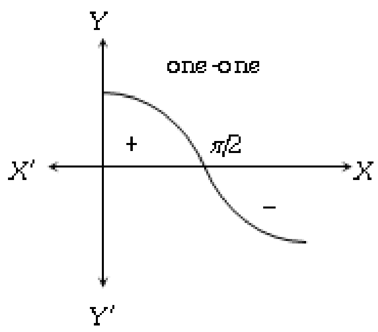
If the domain of the function is in one quadrant then the trigonometric functions are always one-one.

If trigonometric function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many-one.

$f:(0,\pi), f(x) = \sin x$



$f:(0,\pi), f(x) = \cos x$



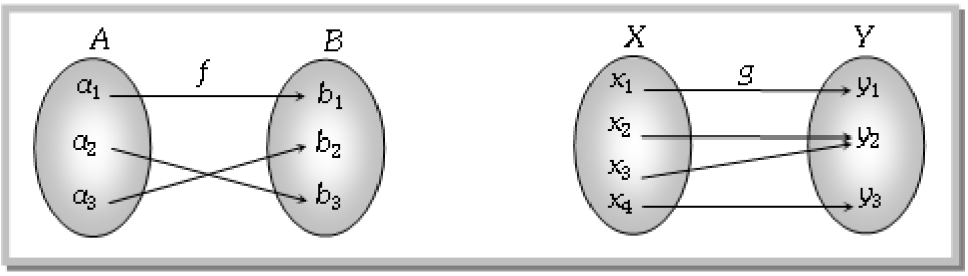
In three consecutive quadrants trigonometric functions are always many-one.

3. **Onto function (surjection)**

A function f: A → B is onto if each element of *B* has its pre-image in *A*. Therefore, if $f^{-1}(y) \in A$,

$\forall\ y \in B$ then function is onto. In other words, Range of *f* = Co-domain of *f*.

e.g. The following arrow-diagram shows onto function.



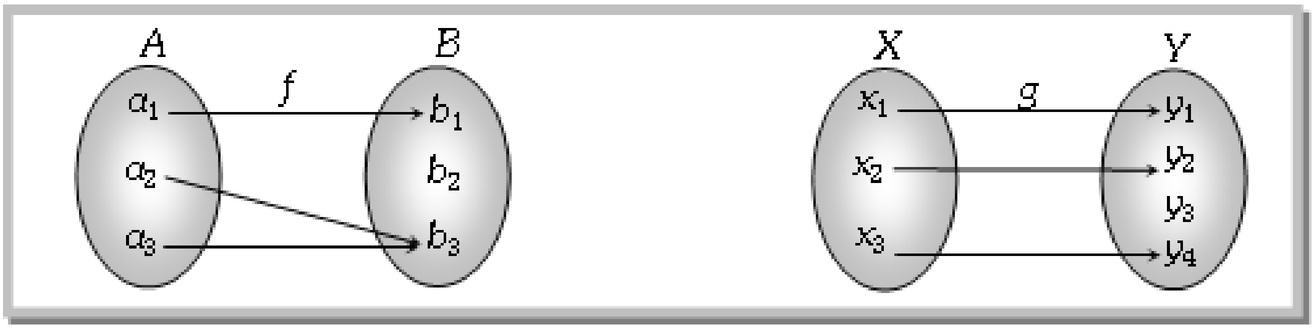
i. **Number of onto function (surjection)** : If *A* and *B* are two sets having *m* and *n* elements respectively such that $1 \le n \le m$, then number of onto functions from *A* to *B* is $\sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r\, r^m$.

4. **Into function**

A function f: A → B is an into function if there exists an element in *B* having no pre-image in *A*.

In other words, f: A → B is an into function if it is not an onto function.

E.*g*. The following arrow-diagram shows into function.



i. **Method to find onto or into function**

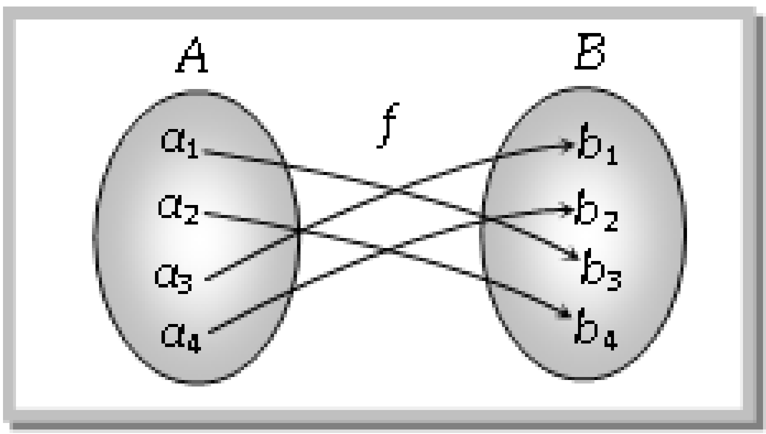
a. If range = co-domain, then f(x) is onto and if range is a proper subset of the co-domain, then f(x) is into.

b. Solve f(x) = y by taking *x* as a function of *y*
i.e., g(y) (say).

c. Now if g(y) is defined for each y ∈ co-domain and g(y) ∈ domain for y ∈ co-domain, then f(x) is onto and if any one of the above requirements is not fulfilled, then f(x) is into.

5. One-one onto function (bijection)

A function f : A → B is a bijection if it is one-one as well as onto.



In other words, a function f : A → B is a bijection if

i. It is one-one i.e.,f(x) = f(y) ⇒x = y for all x, y∈A.

ii. It is onto i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Clearly, *f* is a bijection since it is both injective as well as surjective.

Number of one-one onto function (bijection)

If *A* and *B* are finite sets and $f: A \to B$ is a bijection, then *A* and *B* have the same number of elements. If *A* has *n* elements, then the number of bijection from *A* to *B* is the total number of arrangements of *n* items taken all at a time *i.e.* n!.

6. **Algebraic functions**

Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations $+, -, \times$ and $\div$ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}} + 5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$ (iii) $3x^4 - 5x + 7$

The algebraic functions can be classified as follows:

i. **Polynomial or integral function** : It is a function of the form $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, where $a_0 \neq 0$ and $a_0, a_1, ...a_n$ are constants and $n \in N$ is called a polynomial function of degree *n*

E.*g.* $f(x) = x^3 - 2x^2 + x + 3$ is a polynomial function.

The polynomial of first degree is called a linear function and polynomial of zero degree is called a constant function.

ii. **Rational function**:

The quotient of two polynomial functions is called the rational function.

e.g. $f(x) = \frac{x^2 - 1}{2x^3 + x^2 + 1}$ is a rational function.

iii. **Irrational function:** An algebraic function which is not rational is called an irrational function.

e.g. $f(x) = x + \sqrt{x} + 6,\ g(x) = \dfrac{x^3 - \sqrt{x}}{1 + x^{1/4}}$ are irrational functions.

7. Transcendental function

A function which is not algebraic is called a transcendental function. *e.g.*, trigonometric; inverse trigonometric, exponential and logarithmic functions are all transcendental functions.

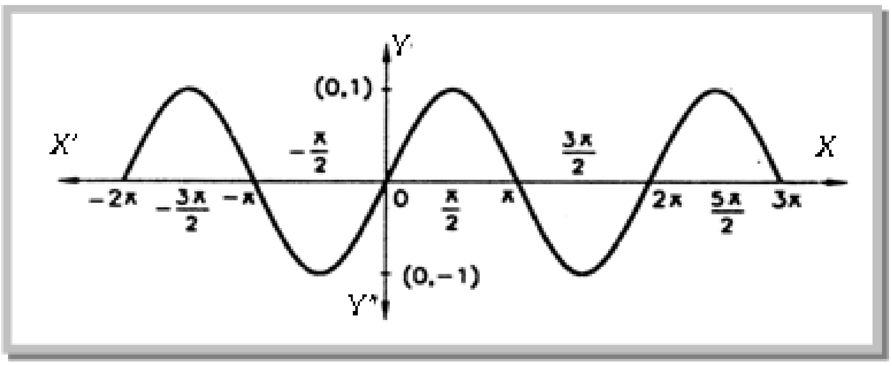
i. Trigonometric functions :

A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.

a. Sine function :

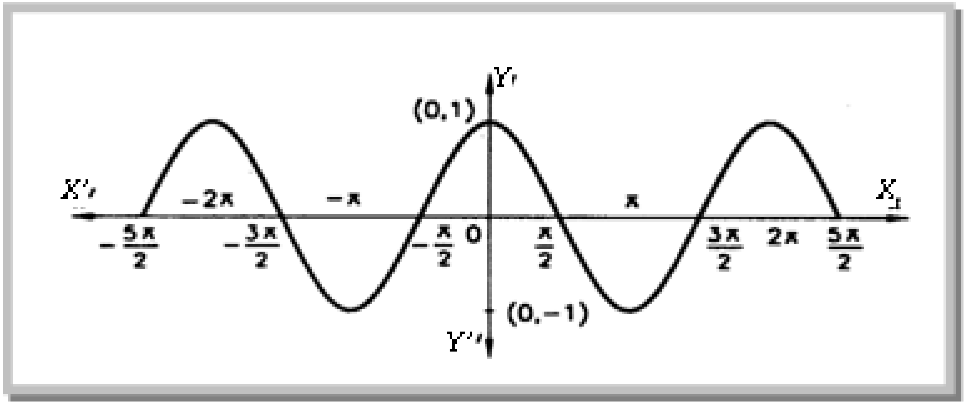
The function that associates to each real numbers *x* to sin x is called the sine function. Here *x* is the radian measure of the angle.

The domain of the sine function is *R* and the range is [-1, 1].



b. Cosine function:

The function that associates to each real number x to cos x is called the cosine function. Here x is the radian measure of the angle. The domain of the cosine function is *R* and the range is [– 1, 1].

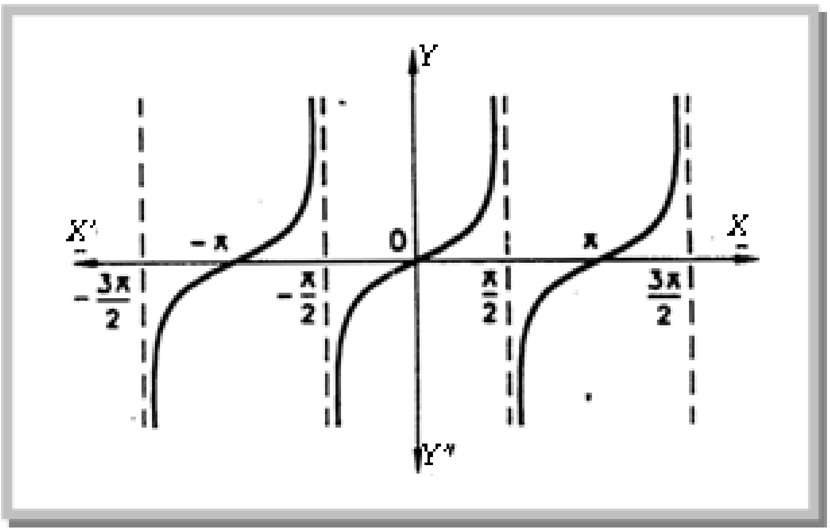


c. Tangent function

The function that associates a real number *x* to tan *x* is called the tangent function.

Clearly, the tangent function is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc. So, the domain of the tangent function is $R - \{(2n+1)\frac{\pi}{2} | n \in I\}$.

Since it takes every value between – ∞ and ∞. So, the range is *R*. Graph of f(x) = tan x is shown in figure.



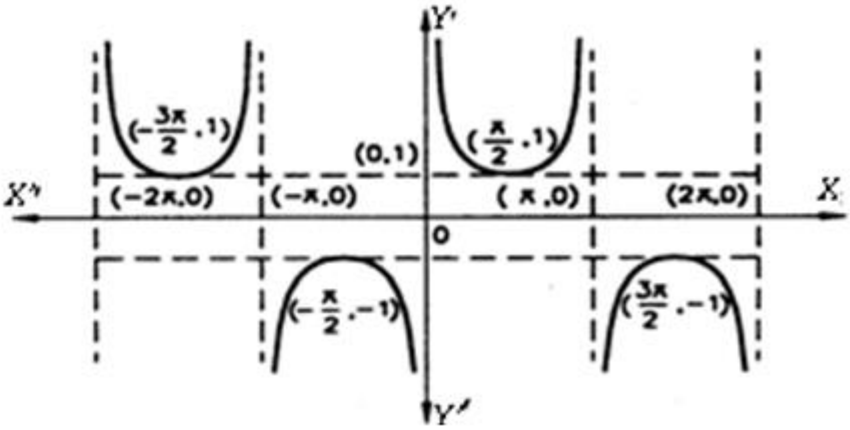
d. Cosecant function

The function that associates a real number *x* to cosec *x* is called the cosecant function.

Clearly, cosec *x* is not defined at x= n π, n ∈ I. *i.e.,* $0, \pm \pi, \pm 2\pi, \pm 3\pi$ etc.

So, its domain is $R - \{n\pi \mid n \in I\}$. Since cosec x ≥ 1 or cosec x ≤ −1. Therefore, range is

(−∞, −1] ∪ [1, ∞). Graph of f(x) = cosec x is shown in figure.



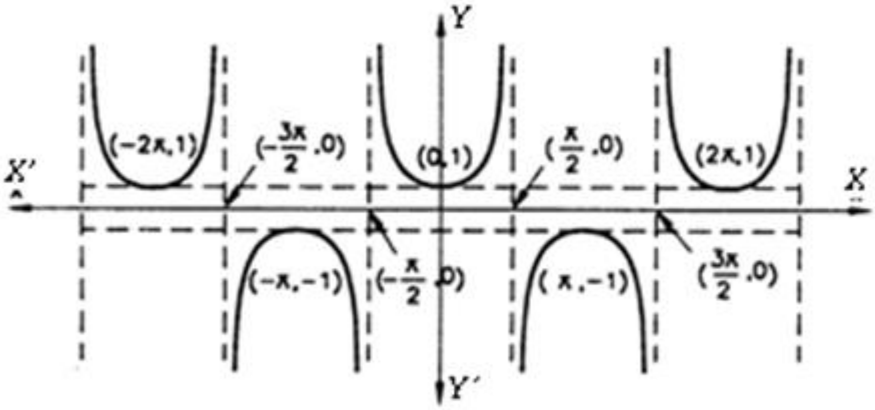
e. Secant function

The function that associates a real number *x* to sec *x* is called the secant function.

Clearly, sec *x* is not defined at odd multiples of $\frac{\pi}{2}$

i.e., $(2\pi + 1)\frac{\pi}{2}$, where n ∈ I. So, its domain is $R - \{(2n+1)\frac{\pi}{2} \mid n \in I\}$. Also, |sec x | ≥ 1, therefore its range is (−∞, −1] ∪ [1, ∞).

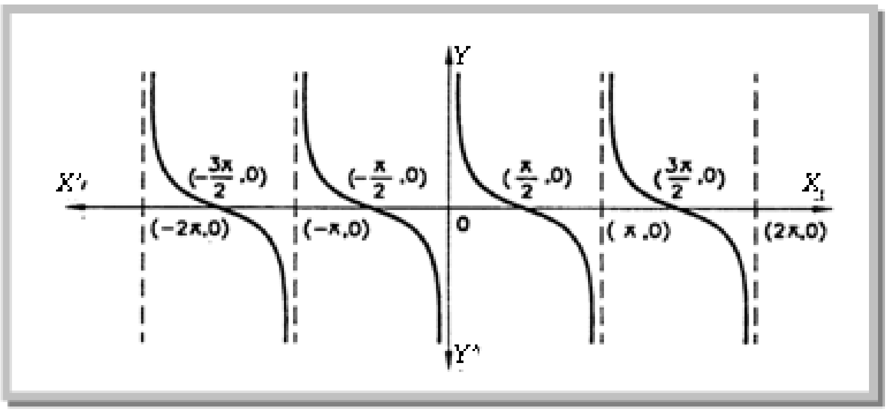
Graph of f (x) = sec x is shown in figure.



f. Cotangent function:

The function that associates a real number *x* to cot *x* is called the cotangent function. Clearly, cot *x* is not defined at x = n π, n∈ I i.e., at $n = 0, \pm p, \pm 2p$ etc.

So, domain of cot x is R- {np | n Î I}. The range of f(x) = cot x is *R* as is evident from its graph in figure.



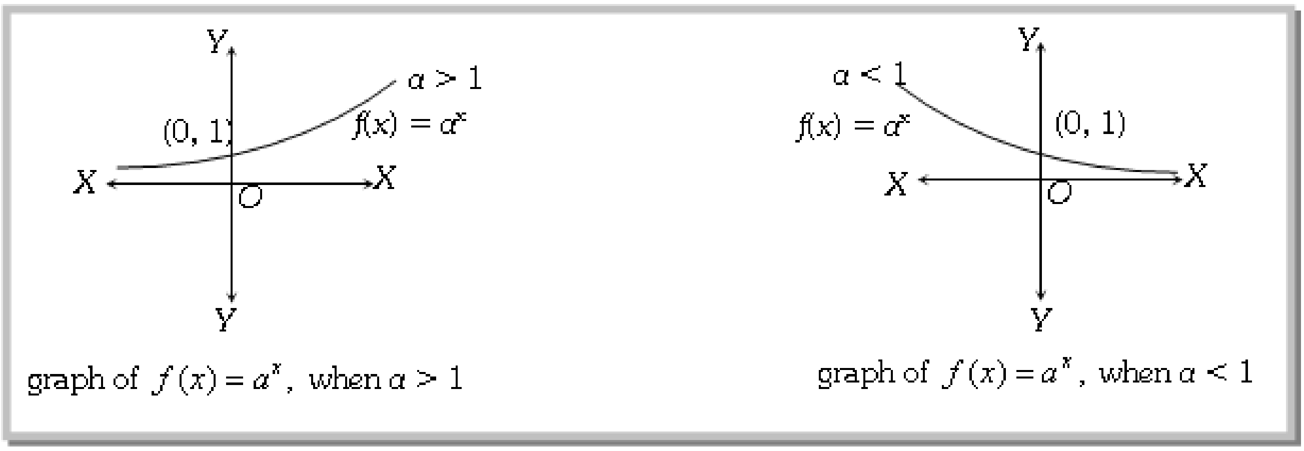
ii. Inverse trigonometric functions

Function	Domain	Range	Definition of the function
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1} x$ Û $x = \sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1} x$ Û $x = \cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y = \tan^{-1} x$ Û $x = \tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y = \cot^{-1} x$ Û $x = \cot y$
$\operatorname{cosec}^{-1} x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$	$y = \operatorname{cosec}^{-1} x$ Û $x = \operatorname{cosec} y$
$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - [\pi/2]$	$y = \sec^{-1} x$ Û $x = \sec y$

iii. Exponential function:

Let $a \neq 1$ be a positive real number.

Then $f: R \to (0, \infty)$ defined by $f(x) = a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$.



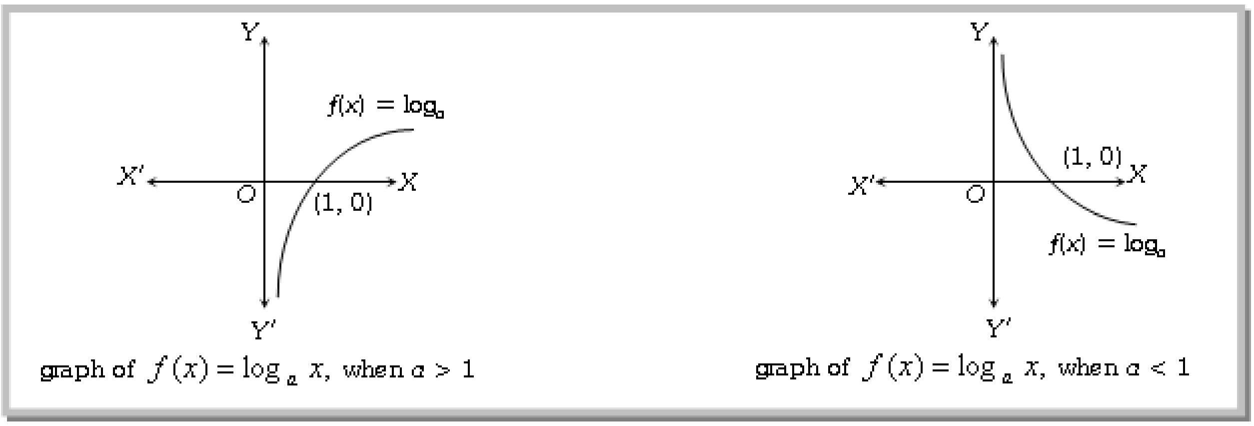
graph of $f(x) = a^x$, when $a > 1$

graph of $f(x) = a^x$, when $a < 1$

iv. **Logarithmic function**:

Let a ≠ 1 be a positive real number.

Then f: (0, ∞) → R defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is (0, ∞) and range is *R*.



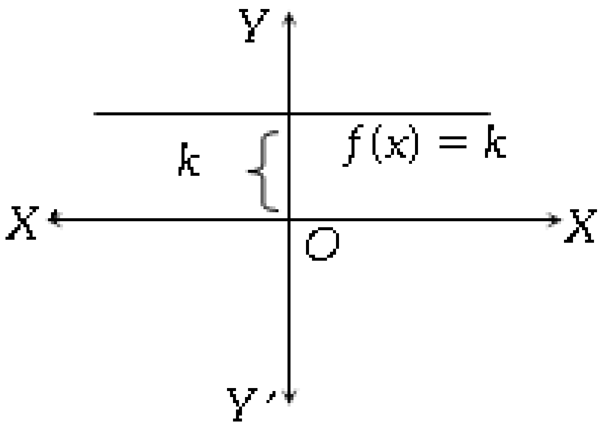
8. **Explicit and implicit functions**

A function is said to be explicit if it can be expressed directly in terms of the independent variable. If the function cannot be expressed directly in terms of the independent variable or variables, then the function is said to be implicit. *e.g.* $y = \sin^{-1} x + \log x$ is explicit function, while

$x^2 + y^2 = xy$ and $x^3y^2 = (a-x)^2(b-y)^2$ are implicit functions.

9. Constant function

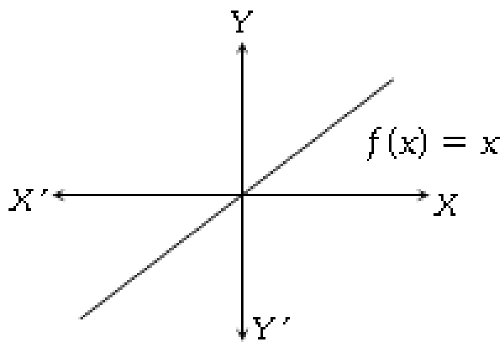
Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in figure and it is above or below the x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.



10. Identity function

The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on R. Clearly, the domain and range of the identity function is R.

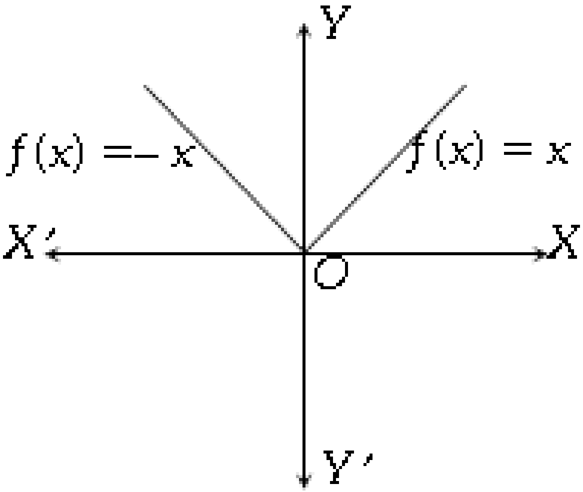
The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of x-axis.



11. Modulus function:

The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function.

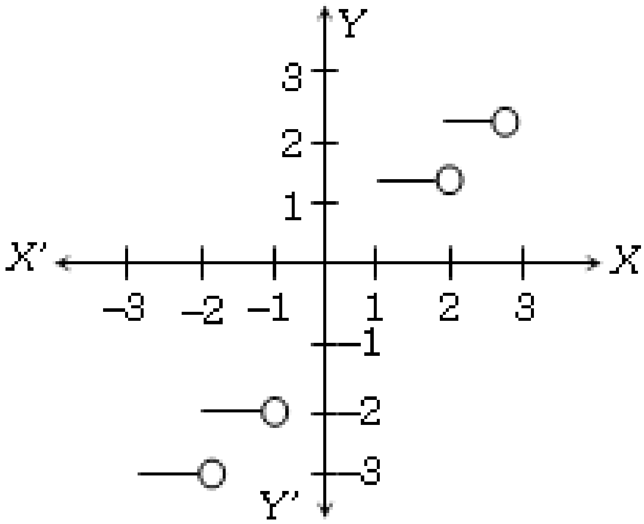
The domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers.



12. Greatest integer function

Let f(x) = [x] where [*x*] denotes the greatest integer less than or equal to *x*. The domain is *R* and the range is I. e.g. [1.1] = 1, [2.2] = 2, [– 0.9] = –1,

[– 2.1] = – 3 etc. The function f defined by f(x) = [x] for all x ∈ R, is called the greatest integer function



13. Fractional part

We know that x ≥ [x]. The difference between the number '*x*' and its integral value '[*x*]' is called the fractional part of *x* and is symbolically denoted as {*x*}. Thus, {x} = x – [x]

e.g., if x = 4.92 then [x] = 4 and {x}= 0.92.

- Fractional part of any number is always non negative and less than one.

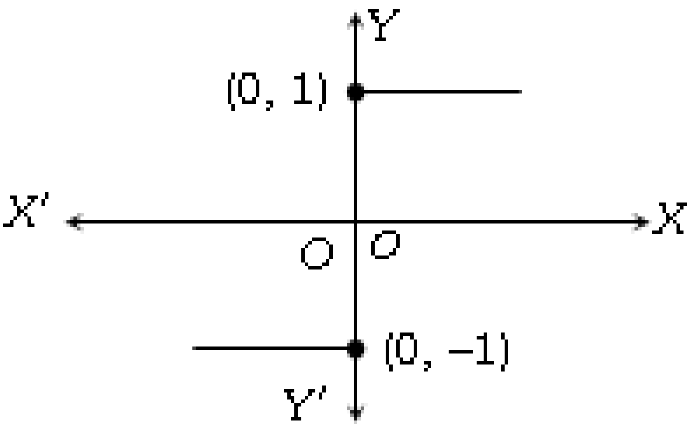
14. Signum function

The function defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases} \text{or} f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$$

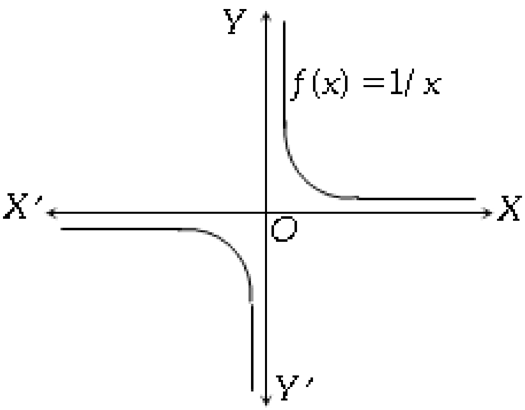
is called the signum function.

The domain is *R* and the range is the set {-1, 0, 1}.



15. Reciprocal function

The function that associates each non-zero real number x to be reciprocal $\frac{1}{\mathrm{x}}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to R-{0} *i.e.,* the set of all non-zero real numbers. The graph is as shown.



Domain and Range of Some Standard Functions

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	{K}
Reciprocal function $\frac{1}{x}$	R_0	R_0
x^2, $\lvert x \rvert$	R	$R^+ \cup \{0\}$
x^3, $x \lvert x \rvert$	R	R
Signum function	R	{−1, 0, 1}
$x + \lvert x \rvert$	R	$R^+ \cup \{0\}$
$x - \lvert x \rvert$	R	$R^- \cup \{0\}$
[x]	R	I
x − [x]	R	[0, 1)
$\sqrt{x}$	$[0, \infty)$	R
a^x	R	R^+
log x	R^+	R
Sin x	R	[−1, 1]
Cos x	R	[−1, 1]
Tan x	$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots\right\}$	R
Cot x	$R - \{0, \pm\pi, \pm 2\pi, \ldots\ldots\}$	R
Sec x	$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots\right\}$	R −(−1, 1)
Cosec x	$R - \{0, \pm\pi, \pm 2\pi, \ldots\ldots\}$	R − (− 1, 1)
$Sin^{-1}x$	[−1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$Cos^{-1}x$	[−1, 1]	[0, π]
$Tan^{-1}x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$Cot^{-1}x$	R	(0, π)
$Sec^{-1}x$	R−(−1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$Cosec^{-1}x$	R−(−1, 1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Even and Odd function

1. Even function

If we put ($-x$) in place of x in the given function and if $f(-x) = f(x)$, $\forall\ x \in$ domain then function $f(x)$ is called even function.

e.g. $f(x) = e^x + e^{-x}, f(x) = x^2, f(x) = x \sin x,\ f(x) = \cos x,$ $f(x) = x^2 \cos x$ all are even function.

2. Odd function

If we put ($-x$) in place of x in the given function and if $f(-x) = -f(x)$, $\forall\ x \in$ domain then $f(x)$ is called odd function. *e.g*

. $f(x) = e^x - e^{-x},\ f(x) = \sin x, f(x) = x^3, f(x) = x \cos x,$ $f(x) = x^2 \sin x$. all are odd function

Periodic Function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x + T) = f(x)$, $\forall\ x \in$ domain. Here the least positive value of T is called the period of the function. Clearly $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots\ldots$.

e.g. sinx , cos x , tan x are periodic functions with period 2 π, 2π and π respectively.

Some standard results on periodic functions

Functions	Periods
(1) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\operatorname{cosec}^n x$	π; if n is even 2 π; if n is odd or fraction
(2) $\tan^n x$, $\cot^n x$	Π; n is even or odd.
(3) $\lvert \sin x \rvert, \lvert \cos x \rvert, \lvert \tan x \rvert, \lvert \cot x \rvert, \lvert \sec x \rvert, \lvert \operatorname{cosec} x \rvert$	π
(4) x - [x]	1
(5) Algebraic functions e.g., $\sqrt{x}, x^2, x^3 + 5, \ldots$.etc	Period does not exist

Composite Function

If $f: A \to B$ and $g: B \to C$ are two function then the composite function of *f* and *g*, gof

$A \to C$ will be defined as $gof(x) = g[f(x)]$, $\forall\ x \in A$

1. **Properties of composite function**

i. f is even, g is even $\Rightarrow$ fog even function.

ii. f is odd, g is odd $\Rightarrow$ fog is odd function.

iii. f is even, g is odd $\Rightarrow$ fog is even function.

iv. f is odd, g is even $\Rightarrow$ fog is even function.

v. Composite of functions is not commutative

i.e., $fog \neq gof$

vi. Composite of functions is associative

i.e., $(fog)oh = fo(goh)$

vii. If $f: A \to B$ is bijection and $g: B \to A$ is inverse of f. Then $fog = I_B$ and $gof = I_A$.

Where, I_A and I_B are identity functions on the sets A and B respectively.

viii. If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijection and $(gof)^{-1} = (f^{-1} og^{-1})$.

ix. $fog \neq gof$ but if , $fog = gof$ then either $f^{-1} = g$

or $g^{-1} = f$ also, $(fog)(x) = (gof)(x) = (x)$

Inverse Functions

If $f: A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each element $b \in B$ with element $a \in A$ such that $f(a) = b$ is called the inverse function of the function $f : A \to B$.

$f^{-1}: B \to A$, $f^{-1}(b) = a \Rightarrow f(a) = b$

In terms of ordered pairs inverse function is defined as $f^{-1} = (b, a)$ if $(a, b) \in f$.

For the existence of inverse function, it should be one-one and onto.

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